ORDINALITY, NEUROSCIENCE AND THE EARLY LEARNING OF NUMBER

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Throughout the twentieth century there was debate as to the primacy of ordinality or cardinality in the development of the concept of number. Psychological experiments have largely given way to neuro-science in deciding this issue. There are results suggesting students' awareness of symbol-symbol relations is the best predictor of future mathematical attainment, which could be interpreted as meaning ordinality is the key awareness. This report draws on evidence from a recent project in primary schools in the UK that took an ordinal approach to learning number. One suggestion arising from this work is the potential educational power of a pedagogy based on developing an awareness of mathematical structure.

INTRODUCTION

In this report, I first set up the theoretical notion of an ordinal approach to learning number, partly drawing on neuro-scientific evidence. I then discuss possible educational implications, before reporting on an empirical study in Primary classrooms in the UK where, although the focus was not on ordinality, it is clear the approach taken to number was ordinal. I conclude with implications for further study.

ORDINALITY VERSUS CARDINALITY

Ordinality refers to the capacity to place numbers in sequence, for example, to know that 4 comes before 5 and after 3 in the sequence of natural numbers. Cardinality refers to the capacity to link numbers to collections, e.g., to know that "4" is the correct representation to denote a group of four objects. A significant question dealt with in the twentieth century, was which aspect of number was most primitive. On the assumption that ordinality and cardinality are the only two dimensions to developing a concept of number, there are three possible views and each one had its proponents. It could be that cardinality is primary (Russell, 1903), it could be that ordinality is primary (Gattegno, 1974), and it could be that both are equally primary (Piaget, 1952). I will briefly summarise each perspective.

Russell (1903) based his analysis of number on the concept of cardinality. For Russell, a number was what was common to sets containing members that could be placed in one-to-one correspondence. Lest there be doubt that questions of mathematical philosophy have relevance, it is only necessary to look at the prevalence of one-to-one mapping tasks in the first years of schooling in the UK, or the fact that work on number is limited to the integers 1-20 (the ones we can grasp), in the early years, to see the influence of Russell's thinking.

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An opposing view is that ordinality is the more primary. This view was used to inspire at least one mathematics curriculum (Gattegno, 1970) and the use of Cuisenaire rods. In Gattegno's curriculum, students' first experiences are to play with the Cuisenaire rods (wooden blocks with 1cm square faces and different lengths – each length associated with a unique colour) and work on relations (bigger than, smaller than). The first number to be introduced is "2", to represent the action of placing two rods of the same size to match the length of a single rod. Numbers are introduced as relations, rather than denoting objects.

A third perspective is that both ordinality and cardinality are equally primitive, and such a view was advocated by Piaget (1952). Piaget believed that the development of ordination and cardination was characterised by the same three stages, which occurred at the same age, hence his conclusion that they are acquired simultaneously.

Experiments in the 1970s appeared to suggest that ordinality occurred in young children at a much earlier age than cardinality (Brainerd, 1979). Recently, the kind of ingenious psychological experiment conducted in the twentieth century, has given way to brain research. One of the findings of broad agreement from neuro-science is that humans share an early (in evolutionary terms) Approximate Number System (ANS), our 'number sense' which we use to judge the relative size of groups of objects (Neider and Dehaene, 2009), i.e., the ANS is a non-symbolic form of numerical reasoning. Research is currently being undertaken to try and map out how the ANS links to our symbolic use of number, since there is evidence that ANS acuity is correlated with later mathematical achievement (e.g., Gilmore et al., 2010).

Some studies suggest a link between our symbolic and non-symbolic awareness of number, which could be taken to imply that cardinality is the key to learning early number. However, the situation may not be as simple as that. Lyons and Beilock (2011) suggest that many experiments related to ANS share an *assumption* that cardinality is the primary aspect of the number concept. Lyons and Beilock (2011, 2013) conducted experiments that test this assumption and their conclusion was:

> a key aspect of transitioning from ANS to symbolic representations of number involves extraction of ordinal information from the ANS and codification of these ordinal relations in terms of direct associations between symbolically represented quantities (2011, p. 257).

In other words, ANS acuity may not be a simple case of awareness of cardinality. Instead, codifying relations between symbols for numbers (characteristic of ordinality) may be key. Furthermore, Lyons and Beilock (2013) found that qualitatively distinct areas of the brain are active during ordinal tasks with number symbols, compared to tasks involving collections of objects (with or without the link to number symbols). There is evidence, then, that in the development of our concept of number, distinct processes are occurring in relation to our awareness of relations between number symbols (in an ordinal sense) and our awareness of how to link objects to numbers. Furthermore, there is evidence (again, from brain imaging) that when working with number in more complex contexts, areas of the brain significant for linking numbers to objects are not activated (Lyons &Beilock, 2011).

There are different interpretations of the neuro-scientific evidence. But one clear hypothesis to emerge is that students' awareness of ordinality may be distinct from awareness of cardinality and, in terms of developing skills needed for success in mathematics, that ordinality is the more significant. If such a conclusion were accepted, it would represent a huge challenge to current practice in the UK where, as stated above, the emphasis in the first years of schooling is firmly on linking number symbols to collections of objects.

In the next section of this report, I draw out educational implications of taking an ordinal approach to number, before then reporting on the results of an empirical study conducted in the UK where such an approach was adopted.

EDUCATIONAL IMPLICATIONS OF AN ORDINAL APPROACH

To take an ordinal approach to number, the focus shifts from linking numbers with the concrete (collections of objects) onto linking numbers with each other. Such an approach was developed by Gattegno (1974) where number is introduced as a relation. Rather than an appeal to collections of objects, number skills and awarenesses can be developed from a structure. As well as the structured Cuisenaire rods, mentioned above, Gattegno devised a chart (see Figure 1) that offers one powerful view of our number system.

0.001	0.002	0.003	0.004	0.005	0.006	0.007	0.008	0.009
0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
1	2	3	4	5	6	7	8	9
10	20	30	40	50	60	70	80	90
100	200	300	400	500	600	700	800	900
1000	2000	3000	4000	5000	6000	7000	8000	9000

Figure 1: Gattegno's tens chart

There is a choice of what rows to display and early work may leave the decimal rows hidden, perhaps with larger numbers added below. When introducing the Gattegno chart to a group for the first time, students need to see how numbers are named on the chart. Rather than concern about the meaning or place value of numbers, the focus is on how to say and write numbers and to gain awareness of how they are ordered. The teacher might tap on a number in the units row and get the class to chant back in unison the number name. This can extends to numbers in the tens row. For example, the teacher taps on "4" (class chant FOUR) and then "40" (class chant FOUR-TY); tap on "6" and then "60"; tap on "8" and then "80". Attention can be focused on how the number name changes (i.e., adding '-ty'), the task for students is to say and read the numbers. In contrast to limiting students to 1-20, on such an approach, the single awareness of how to move from the units to tens row allows access to 1-99 (students can enjoy saying the structurally correct “three-ty” for thirty, “two-ty” for twenty and “one-ty” for ten).

Gattegno’s ordinal approach to number was the background to a research project in the UK that aimed to develop creativity in the Primary mathematics curriculum as a way of tackling underachievement. In the next section I report on this project, drawing out the links to an ordinal approach (that were not made explicit at the time), before giving some results and offering implications.

AN EMPIRICAL STUDY IN THE UK

During 2010-2013, I worked in collaboration with the charity “5x5x5=creativity” (5x5x5) with 5 different Primary schools (and one teacher in each school) to develop creative approaches to teaching mathematics. Projects with 5x5x5 often involve an artist working with a group of students in a school, to develop and document their learning in relation to a provocation. In 2010-11, 2011-12 and 2012-13, I acted as a mathematician-artist with the project schools as well as co-ordinating meetings (six a year) between teachers from project classrooms. As the mathematician-artist, I would go in to schools to take lessons. In all schools, we agreed the project lessons would centre around the notion of students ‘becoming a mathematician’. We emphasised that mathematicians look for pattern and ask questions. The content of the lessons I taught was always discussed and agreed with the classroom teachers and we would de-brief afterwards. Teachers continued to work on developing activities that would allow students to notice and develop patterns, when I was not there, and at the meetings would share their ideas and activities (see Coles, Fernandez and Brown, 2013). Some teachers devoted one lesson a week to activities linked to ‘becoming a mathematician’, in a few cases, teachers shifted their entire approach to teaching mathematics and every lesson had a focus on students’ noticing and emerging ideas.

The tool that was used more than any other in schools (in the context of the project) was the Gattegno chart (Figure 1). A common activity with year 1 (age 5-6) students was to tap on a number of the chart and get the class to chant back (in unison) the number one higher (or one lower). After working on this and taking different starting points, the students might be invited to choose their own starting number and to keep on either adding or subtracting 1 and to see what they noticed.

Another activity tried in several schools, usually with year 3 or 4 students (age 7-9), involved tapping on the chart and getting students to chant back the number ten times bigger. This can be done on the chart with a simple movement down a row. After practising in unison, the class do the same for division by 10, then for multiplication and division by 100. For this activity, the class were then invited to choose a ‘starting number’ somewhere on the chart, to go on a ‘journey’ of multiplying and dividing by powers of 10, with the challenge to get back where they started from.

To give a sense of what students might do in the course of these activities, a typical example from a students' book is copied below (see Figure 2). This student was in year 3 (age 8) and was working on multiplication and division journeys. She had decided to challenge herself to go on a journey and get back in one go, "I went back in one" is her comment on the right of the page (Figure 2).

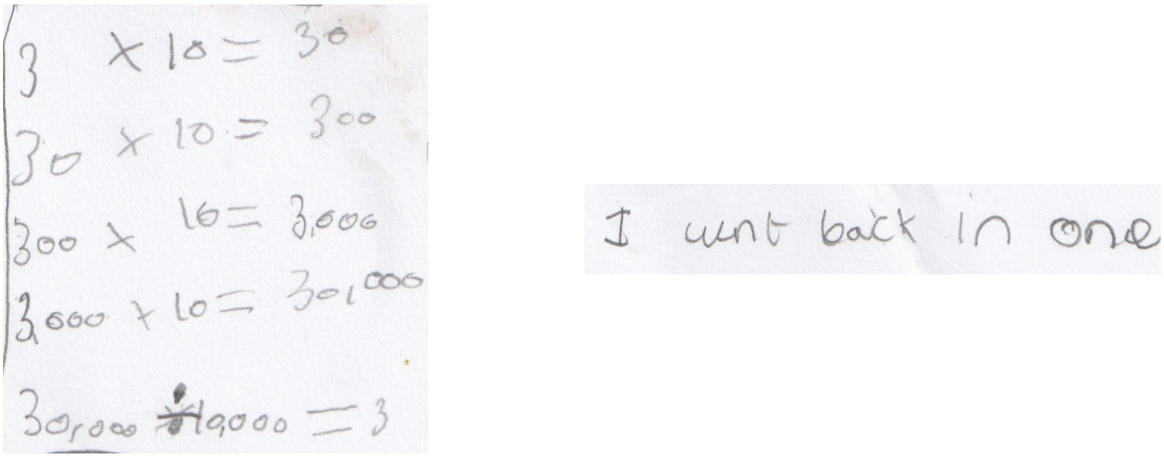

Figure 2: One student's 'journey' and comment

Division by 10,000 is many years in advance of what a year 3 students would normally be expected to compute. There is tentative evidence in Figure 2 that this student has become aware of a relation between successive multiplications by 10 and their inverse. She is making connections between the symbols themselves and seems to be gaining some confidence in working with symbols in their own right (something closely linked to an ordinal view of number).

While ordinality was not an explicit focus of the project, the description above, of activities on the Gattegno chart, demonstrates that the approach to number was one of linking symbols to symbols and moving away from concrete representations.

METHODOLOGY

The original focus of the project was on teacher development, hence audio recordings were taken of all meetings with teachers and these have been analysed (e.g., Coles, Fernandez and Brown, 2013). For this report, I have re-analysed the audio recordings of teacher meetings from 2012-13, using the theoretical framing of ordinality/cardinality, i.e., looking out for instances where ordinality/cardinality was being discussed as an issue. The taking of multiple views of data is in keeping with the enactivist methodology (Reid, 1996) that underpinned the study. Rough transcripts had already been created for the project meetings. I re-read these transcripts and returned to the audio data to confirm and make accurate the transcription of any sequence of talk that touched on issues of symbol use or the connection between symbols and objects. I also report briefly on the statistical progress data that was collected across the 5 schools. All the schools routinely monitored student progress (in relation to a system of National Curriculum levels) and schools, at points throughout the year, assessed students from project classrooms. Assessments were made by teachers, informed by written tests and moderated by a local authority. For the purposes of the project, progress was judged from the end of the year before the work began, to the end of the year in which the project took place.

RESULTS

The issue of ordinality/cardinality is raised at three meetings during 2012-13, and always by Teacher G. These three meetings are reported briefly below and analysed.

In November 2012, Teacher G (who had a year 2, age 6-7, class) reflected on the work of a student who is attaining well below government expectations for his age.

> G: He's loved doing the number journeys, loved exploring what's happening when dividing by ten and dividing by a hundred. He didn't always know what the numbers were. He might know it has two zeros at the end but not know it's six hundred. He's used the pattern in terms of how it looks without being able to say the number. That makes me a bit uneasy.

The student in question appears to have been able to write out some journeys successfully, but G expresses concern that he is working with numbers he cannot say. A similar discomfort was expressed again when Teacher G reflected (February 2013) on further work he was doing using the Gattegno chart and a group of students who had been working on writing out multiples of 21 (students had chosen what multiples to work on):

> G: They were doing 21 and then 42 and 63 and 84 and they were looking to see what was happening with the digits. So they could see what was happening and they could see could see the pattern, they could predict next one … I'm not sure if it's a danger but I'm aware some children see the patterns and can write a sequence of digits but maybe not know how to read those digits as a number … it just makes me aware you can't just leave it there because they just see it as patterns of numbers and they don't get to feel the truth underneath it, the place value underneath it.

I interpret Teacher G here as grappling with precisely the ordinal/cardinal issue. He reports his students being successful writing multiples of 21 (beyond what would be expected of students at that age in the UK) and yet being concerned whether students got the 'truth underneath it', the place value sense of the number – which may be a wish for a more cardinal awareness of the link to objects.

Another teacher responded directly after G's turn above:

> E: you mean the symbols representing numbers have become disconnected from what they represent … the thing we're always trained not to do is to take children beyond those numbers they can grapple and handle. It's almost the whole thing is, what happens when we do do that, and is it empowering or is it actually quite shocking, quite weird, I don't know.

Teacher E here interprets the whole purpose of the 5x5x5 project: 'the whole thing is, what happens when we do' take children beyond those numbers they can 'grapple and handle. E is a headteacher and he gives an interesting insight into the orthodoxy of Primary teacher training in the UK: 'the thing we're always trained *not* to do', is move beyond students' cardinal sense of number.

By July 2013, students in G's class made, on *average*, 18 months progress over the academic year. The headteacher at G's school described the impact of the project as 'transformational' and in 2012 and 2013 (the years the school was involved) the school achieved its best ever results for the end of year 2 (the project class in 2012 and 2013). The progress by students in this school was higher than in the other 4 project schools (although in all schools, student progress matched or exceeded government expectations). Factors that were different at G's school compared to the others included: the teacher involved in 2011-12 having responsibility for developing numeracy across the school; the teachers at this school in 2011-12 and 2012-13 adopting a 'project' approach more consistently throughout their teaching than in other schools; students in the school having lower prior attainment than other schools and coming from areas of higher deprivation (as judged by the UK school inspectorate, Ofsted).

Teacher G and E's concerns and questions are significant and also give an insight into the challenge of creating new ways of working. Teacher G was subject to an Ofsted inspection during 2013, which he discussed at the meeting in June 2013. Whilst being impressed by what they saw, the inspector picked up on the issue of students working with numbers they could not read and raised this as a concern. The issue of reading numbers is an intriguing one. The Gattegno chart (Figure 1) can be used to support number reading and can be powerful in this respect. I interpret, in the concerns expressed by G, E or the inspector the exact issues discussed at the start of this report – what is a number? and, what does it mean to know a number? At what point is it okay to work with numbers we cannot 'grapple with and handle'?

DISCUSSION

In this report, I have presented neuro-scientific evidence and results from an empirical study that both suggest the idea of an ordinal approach to early number should at least be taken seriously as a possible focus in Primary school. Experimental brain studies have suggested that awareness of ordinality may be the key attribute determining the chance of success in later mathematics. In the empirical study, we certainly witnessed students becoming excited, interested and successful in mathematics, through a focus on the structure of the number system and through giving students permission to explore larger numbers than they would normally be allowed. There is clearly a need for further work on the neuro-scientific basis of early number acquisition and this is on going. There is also a need for further work in the classroom, and with teachers, to develop and trial materials, activities and ways of working that support students' awareness of ordinality. Not only that, we need to know more about effective way of working with teachers to support the development of an ordinal approach to number and to address the real concerns expressed about place value.

There are also possible implications for mathematics teaching in higher years. One interpretation of the success of an ordinal approach to early number is that it stems from having a focus on developing awareness of mathematical structure in an almost game-like manner. Once the structure (the rules of the game) is established (for example, through choral response with the Gattegno chart, Figure 1) there is space for creativity as students enter into a dialogue with the challenge of learning mathematics. There is nothing to stop such an approach being used at any level of mathematics (see Coles & Brown, 2013).

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